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Research Article

An Exponentially Fitted Method for Singularly Perturbed Delay Differential Equations

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This paper deals with singularly perturbed initial value problem for linear first-order delay differential equation. An exponentially fitted difference scheme is constructed in an equidistant mesh, which gives first-order uniform convergence in the discrete maximum norm. The difference scheme is shown to be uniformly convergent to the continuous solution with respect to the perturbation parameter. A numerical example is solved using the presented method and compared the computed result with exact solution of the problem.

1. Introduction

Delay differential equations play an important role in the mathematical modelling of various practical phenomena in the biosciences and control theory. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay term [1–4]. Such problems arise frequently in the mathematical modelling of various practical phenomena, for example, in the modelling of several physical and biological phenomena like the optically bistable devices [5], description of the human pupil-light reflex [6], a variety of models for physiological processes or diseases and variational problems in control theory where they provide the best, and in many cases the only realistic simulation of the observed phenomena [7].

It is well known that standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to these problem, whose accuracy does not depend on the parameter value ε, that is, methods that are uniformly convergent with respect to the perturbation parameter [8–10]. One of the simplest ways to derive such methods consists of using an exponentially fitted difference scheme (see, e.g., [10] for motivation for this type of mesh), which are constructed a priori and depend of the parameter ε, the problem data and the number of corresponding mesh points. In the direction of numerical treatment for first-order singularly perturbed delay differential equations, several can be seen in [4, 7, 11].

In order to construct parameter-uniform numerical methods, two different techniques are applied. Firstly, the numerical methods of exponential fitting type (fitting operators) (see [9]), which have coefficients of exponential type adapted to the singular perturbation problems. Secondly, the special mesh approach (see [11, 12]), which constructs meshes adapted to the solution of the problem.

In the works of Amiraliyev and Erdogan [11], special meshes (Shishkin mesh) have been used. The method that we propose in this paper uses exponential fitting schemes, which have coefficients of exponential type.

2. Statement of the Problem

Consider a model problem for the initial value problems for singularly perturbed delay differential equations with delay in the interval $\overline{I} = [0,T]$:

$$\begin{aligned} \varepsilon u'(t) + a(t)u(t) + b(t)u(t-r) &= f(t), \quad t \in I, \\ u(t) &= \varphi(t), \quad t \in I_0, \end{aligned} \tag{2.1}$$

where $I = (0,T] = \bigcup_{p=1}^{m} I_p$, $I_p = \{t : r_{p-1} < t \le r_p\}$, $1 \le p \le m$ and $r_s = sr$, for $0 \le s \le m$ and $I_0 = [-r, 0]$ (for simplicity we suppose that T/r is integer). $0 < \varepsilon \le 1$ is the perturbation parameter, $a(t) \ge \alpha > 0$, $b(t)$, $f(t)$, and $\varphi(t)$ are given sufficiently smooth functions satisfying certain regularity conditions to be specified and r is a constant delay. The solution $u(t)$ displays in general boundary layers at the right side of each points $t = r_s (0 \le s \le m)$ for small values of ε.

In this paper, we present the completely exponentially fitted difference scheme on the uniform mesh. The difference scheme is constructed by the method of integral identities with the use of exponentially basis functions and interpolating quadrature rules with weight and remainder terms integral form [10]. This method of approximation has the advantage that the schemes can also be effective in the case when the continuous problem is considered under certain restrictions.

In the present paper, we analyze a fitted difference scheme on a uniform mesh for the numerical solution of the problem (2.1). In Section 2, we describe the problem. In Section 3, we state some important properties of the exact solution. In Section 4, we construct a numerical scheme for solving the initial value problem (2.1) based on an exponentially fitted difference scheme on a uniform mesh. In Section 5, we present the error analysis for approximate solution. Uniform convergence is proved in the discrete maximum norm. A numerical example in comparison with their exact solution is being presented in Section 6. The approach to construct discrete problem and error analysis for approximate solution is similar to those ones from [10, 11].

Notation. Throughout the paper, C will denote a generic positive constant (possibly subscripted) that is independent of ε and of the mesh. Note that C is not necessarily the same at each occurrence.

3. The Continuous Problem

Here, we show some properties of the solution of (2.1), which are needed in later sections for the analysis of appropriate numerical solution. Let, for any continuous function g, $\|g\|_{\infty,I}$ denotes a continuous maximum norm on the corresponding interval.

Lemma 3.1. *Let $a, b, f \in C^1(\bar{I})$, $\varphi \in C^1(\bar{I}_0)$. Then, for the solution $u(t)$ of the problem* (2.1) *the following estimates hold*

$$\|u(t)\|_{\infty,I_p} \le C_p, \quad 1 \le p \le m, \tag{3.1}$$

where

$$\begin{aligned} C_1 &= \alpha^{-1}\|f\|_{\infty,I_1} + \left(1 + \alpha^{-1}\|b\|_{\infty,I_1}\right)\|\varphi\|_{\infty,I_0}, \\ C_p &= \alpha^{-1}\|f\|_{\infty,I_p} + \left(1 + \alpha^{-1}\|b\|_{\infty,I_p}\right)C_{p-1}, \quad p = 2, 3, \ldots, m. \end{aligned} \tag{3.2}$$

Proof. see [11]. □

4. Discretization and Mesh

In this section, we construct a numerical scheme for solving the initial value problem (2.1) based upon an exponential fitting on a uniform mesh.

We denote by $\overline{\omega}_{N_0}$ the uniform mesh on $\bar{I}$:

$$\overline{\omega}_{N_0} = \left\{ t_i = i\tau,\ i = 0, 1, 2, \ldots, N_0;\ \tau = \frac{r}{N},\ pN = N_0 \right\}, \tag{4.1}$$

which contains N mesh points at each subinterval I_p $(1 \le p \le m)$:

$$\omega_{N,p} = \left\{ t_i : (p-1)N + 1 \le i \le pN \right\}, \quad 1 \le p \le m, \tag{4.2}$$

and consequently

$$\omega_{N_0} = \bigcup_{p=1}^{m} \omega_{N,p}. \tag{4.3}$$

To simplify the notation, we set $g_i = g(t_i)$ for any function $g(t)$, and moreover y_i denotes an approximation of $u(t)$ at t_i. For any mesh function $\{w_i\}$ defined on ϖ_{N_0}, we use

$$\begin{aligned} w_{\bar{t},i} &= \frac{w_i - w_{i-1}}{\tau}, \\ \|w\|_{\infty,N,p} &= \|w\|_{\infty,\omega_{N,p}} := \max_{(p-1)N \le i \le pN} |w_i|, \quad 1 \le p \le m. \end{aligned} \tag{4.4}$$

The approach of generating difference methods through integral identity

$$\chi_i\tau^{-1}\int_{t_{i-1}}^{t_i} Lu(t)\varphi_i(t)dt = \chi_i\tau^{-1}\int_{t_{i-1}}^{t_i} f(t)\varphi_i(t)dt, \tag{4.5}$$

with the exponential basis functions

$$\varphi_i(t) = \exp\left(-\frac{a_i(t_i - t)}{\varepsilon}\right), \quad t_{i-1} \le t \le t_i, \tag{4.6}$$

where

$$\chi_i = \left(\tau^{-1}\int_{t_{i-1}}^{t_i}\varphi_i(t)dt\right)^{-1} = \frac{a_i\rho}{1 - \exp(-a_i\rho)}, \quad \rho = \frac{\tau}{\varepsilon}. \tag{4.7}$$

We note that function $\varphi_i(t)$ is the solution of the problem

$$\begin{gathered} -\varepsilon\varphi_i'(t) + a_i\varphi_i(t) = 0, \quad t_{i-1} \le t < t_i, \\ \varphi_i(t_i) = 1. \end{gathered} \tag{4.8}$$

The relation (4.5) is rewritten as

$$\chi_i\tau^{-1}\varepsilon\int_{t_{i-1}}^{t_i} u'(t)\varphi_i(t)dt + a_i\chi_i\tau^{-1}\int_{t_{i-1}}^{t_i} u(t)\varphi_i(t)dt + b_i\chi_i\tau^{-1}\int_{t_{i-1}}^{t_i} u(t-r)\varphi_i(t)dt + R_i = f_i, \tag{4.9}$$

with the remainder term

$$\begin{aligned} R_i &= R_i^{(1)} + R_i^{(2)} + R_i^{(3)}, \\ R_i^{(1)} &= \chi_i\tau^{-1}\int_{t_{i-1}}^{ti} [a(t) - a(t_i)]u(t)\varphi_i(t)dt, \\ R_i^{(3)} &= \chi_i\tau^{-1}\int_{t_{i-1}}^{ti} [b(t) - b(t_i)]u(t-r)\varphi_i(t)dt, \\ R_i^{(3)} &= \chi_i\tau^{-1}\int_{t_{i-1}}^{ti} [f(t_i) - f(t)]\varphi_i(t)dt. \end{aligned} \tag{4.10}$$

Taking into account (4.5) and using interpolating rules with the weight (see [10]), we obtain the following relations:

$$\varepsilon\theta_i u_{\bar{t},i} + a_i u_i + b_i u_{i-N} + R_i = f_i, \quad 1 \le i \le N_0, \tag{4.11}$$

where

$$\theta_i = 1 + \chi_i \tau^{-1} a_i \varepsilon^{-1} \int_{t_{i-1}}^{t_i} (t - t_i) \varphi_i(t) dt, \tag{4.12}$$

and a simple calculation gives us

$$\theta_i = \frac{a_i \rho}{1 - \exp(-a_i \rho)} \exp(-a_i \rho). \tag{4.13}$$

As a consequence of the (4.11), we propose the following difference scheme for approximation (2.1):

$$\begin{gathered} L y_i := \varepsilon \theta_i y_{\bar{t},i} + a_i y_i + b_i y_{i-N} = f_i, \quad 1 \le i \le N_0, \\ y_i = \varphi_i, \quad -N \le i \le 0, \end{gathered} \tag{4.14}$$

where θ_i is defined by (4.13).

5. Analysis of the Method

To investigate the convergence of the method, note that the error function $z_i = y_i - u_i$, $0 \le i \le N_0$, is the solution of the discrete problem

$$\begin{gathered} \varepsilon \theta_i z_{\bar{t},i} + a_i z_i + b_i z_{i-N} = R_i, \quad 1 \le i \le N_0, \\ z_i = \varphi_i, \quad -N \le i \le 0. \end{gathered} \tag{5.1}$$

where R_i and θ_i are given by (4.10) and (4.13), respectively.

Lemma 5.1. *Let y_i be approximate solution of* (2.1)*. Then the following estimate holds*

$$\|y\|_{\infty,\overline{\omega}_{N,p}} \le \|\varphi\|_{\infty,\omega_{N,0}} Q_p + \alpha^{-1} \sum_{k=1}^{p} \|f\|_{\infty,\omega_{N,k}} Q_{p-k}, \quad 1 \le p \le m, \tag{5.2}$$

where

$$Q_{p-k} = \begin{cases} 1, & \text{for } k = p, \\ \displaystyle\prod_{s=k+1}^{p} \left(1 + \alpha^{-1} \|b\|_{\infty,I_s}\right), & \text{for } 0 \le k \le p-1. \end{cases} \tag{5.3}$$

Proof. The proof follows easily by induction in p. □

Lemma 5.2. *Let z_i be solution of* (5.1)*. Then following estimate holds*

$$\|z\|_{\infty,N,p} \leq C\sum_{k=1}^{p}\|R\|_{\infty,\omega_{N,k}}. \tag{5.4}$$

Proof. It evidently follows from (5.2) by taking $\varphi \equiv 0$ and $f \equiv R$. □

Lemma 5.3. *Under the above assumptions of Section 2 and Lemma 3.1, for the error function R, the following estimate holds*

$$\|R\|_{\infty,\omega_{N},p} \leq C\tau, \quad 1 \leq p \leq m. \tag{5.5}$$

Proof. To this end, it suffices to establish that the functions $R_i^{(k)}$ $(k = 1,2,3)$, involved in the expression for R_i, admit the estimate

$$\|R^{(k)}\|_{\infty,\omega_{N},p} \leq C\tau, \quad k = 1,2,3. \tag{5.6}$$

Using the mean value theorem, we get

$$\begin{aligned}\left|a(t) - a(t_i)\right| &= \left|a'(\xi)(t - t_i)\right|,\\ &= \max_{\omega_{N,p}}\left|a'(\xi)\right|\left|t - t_i\right| \leq C\tau, \quad \xi \in \left[t_{i-1}, t_{i+1}\right].\end{aligned} \tag{5.7}$$

Hence

$$\left|R_i^{(1)}\right| \leq C\tau\tau^{-1}\int_{t_{i-1}}^{t_i}\left|u(t)\right|\varphi_i(t)dt, \tag{5.8}$$

and taking also into account that $0 \leq \varphi_i(t) \leq 1$ and using Lemma 3.1, we have

$$\left\|R^{(1)}\right\|_{\infty,\omega_{N},p} \leq C\tau. \tag{5.9}$$

For $R_i^{(2)}$, in view of $b \in C^1(\bar{I})$ and using Lemma 3.1, we obtain

$$\left|R_i^{(2)}\right| \leq \tau^{-1}\int_{t_{i-1}}^{t_i}\left|b(t) - b(t_i)u(t-r)\right|\varphi_i(t)dt \leq C\int_{t_{i-1}}^{t_i}\left|u(\xi - r)\right|d\xi. \tag{5.10}$$

Hence

$$\left\|R^{(2)}\right\|_{\infty,\omega_{N},p} \leq C\int_{t_{i-1}}^{t_i}\left|u(\xi - r)\right|d\xi, \tag{5.11}$$

and after replacing $s = \xi - r$ this reduces to

$$\|R^{(2)}\|_{\infty,\omega_N,p} \le C\int_{t_{i-1}-r}^{t_i-r} |u(s)|ds = C\left(\int_{-r}^{0} |\varphi(s)|ds + \int_{t_{i-1}}^{t_i} |u(s)|ds\right), \tag{5.12}$$

which yields

$$\|R^{(2)}\|_{\infty,\omega_N,p} \le C\tau\left(\|\varphi\|_{1,0} + C_p\right) = O(\tau). \tag{5.13}$$

The same estimate is obtained for $R_i^{(3)}$ in the similar manner as above. □

Combining the previous lemmas we get the following final estimate, that is, uniformly convergent estimate.

Theorem 5.4. *Let u be the solution of* (2.1) *and y be the solution of* (4.14)*. Then the following estimate holds*

$$\|y - u\|_{\infty,\overline{\omega}_{N,p}} \le C\tau, \quad 1 \le p \le m. \tag{5.14}$$

6. Numerical Results

We begin with an example from Driver [2] for which we possess the exact solution.

$$\begin{gathered} \varepsilon u'(t) + u(t) = u(t-1), \quad t \in [0,T], \\ u(t) = 1 + t, \quad -1 \le t \le 0. \end{gathered} \tag{6.1}$$

The exact solution for $0 \le t \le 2$ is given by

$$u(t) = \begin{cases} -\varepsilon + t + (1+\varepsilon)e^{-t/\varepsilon}, & t \in [0,1], \\ -1 - 2\varepsilon + t + (1+\varepsilon)e^{-t/\varepsilon} + \left[\varepsilon - \dfrac{1}{\varepsilon} + \left(1 + \dfrac{1}{\varepsilon}\right)t\right]e^{(1-t)/\varepsilon}, & t \in (1,2]. \end{cases} \tag{6.2}$$

We define the computed parameter-uniform maximum error $e_\varepsilon^{N,p}$ as follows:

$$e_\varepsilon^{N,p} = \|y - u\|_{\infty,\omega_{N,p}}, \quad p = 1,2, \tag{6.3}$$

where y is the numerical approximation to u for various values of N, ε. We also define the computed parameter-uniform convergence rates for each N:

$$r^{N,p} = \ln\frac{e^{N,p}/e^{2N,p}}{\ln 2}, \quad p = 1,2. \tag{6.4}$$

The values of ε for which we solve the test problem are $\varepsilon = 2^{-i}$, $i = 1,2,\ldots,8$.

Table 1: Maximum errors $e_\varepsilon^{N,1}$ and convergence rates $r^{N,1}$ on $\omega_{N,1}$.

ε	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
2^{-1}	0.0033688	0.0016866	0.000843849	0.000422062	0.000211065
	0.998	0.999	0.999	0.999	
2^{-2}	0.00381473	0.00191236	0.000957428	0.000479026	0.000239591
	0.996	0.996	0.998	0.999	
2^{-3}	0.00386427	0.00194230	0.000973693	0.000487882	0.000243900
	0.992	0.996	0.998	0.999	
2^{-4}	0.00382489	0.00193278	0.000971476	0.00048701	0.000243823
	0.984	0.992	0.996	0998	
2^{-5}	0.00374366	0.00191245	0.000966391	0.000485738	0.000243505
	0.969	0.984	0.992	0.996	
2^{-6}	0.00358208	0.00187183	0.000956223	0.000433195	0.000242869
	0.936	0.969	0.984	0.992	
2^{-7}	0.00326581	0.00179104	0.000935915	0.000477811	0.000241598
	0.866	0.936	0.969	0.984	
2^{-8}	0.00268346	0.0016329	0.00895519	0.00467957	0.000239057
	0.716	0.866	0.936	0.969	

Table 2: Maximum errors $e_\varepsilon^{N,2}$ and convergence rates $r^{N,2}$ on $\omega_{N,2.}$

ε	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
2^{-1}	0.00319858	0.00164347	0.000832995	0.000419339	0.000211065
	0.960	0.980	0.990	0.995	
2^{-2}	0.00600293	0.00300639	0.00150442	0.000752515	0.000376334
	0.997	0.999	0.999	1.00	
2^{-3}	0.00780800	0.00396966	0.00200100	0.00100461	0.000503328
	0.975	0.988	0.994	0.997	
2^{-4}	0.0185227	0.00951902	0.00482057	0.00242576	0.001216820
	0.960	0.981	0.990	0995	
2^{-5}	0.0388137	0.0202932	0.0103797	0.00525228	0.002641280
	0.935	0.967	0.9982	0.9916	
2^{-6}	0.0747962	0.0405973	0.0211784	0.0108201	0.005461600
	0.881	0.938	0.968	0.984	
2^{-7}	0.131822	0.0765885	0.0414891	0.0216210	0.011040200
	0.783	0.884	0.940	0.969	
2^{-8}	0.149561	0.133579	0.0774847	0.0419350	0.021842300
	0.163	0.785	0.885	0.941	

These convergence rates are increasing as N increases for any fixed ε. Tables 1 and 2 thus verify the ε-uniform convergence of the numerical solutions and the computed rates are in agreement with our theoretical analysis.

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